1.) A damped oscillator satisfies $m\ddot{x} + b\dot{x} + kx = 0$, where $F_{\text{damp}} = -b\dot{x}$ is the damping force.

a.) Find the rate of change of the energy $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ (by straightforward differentiation)

b.) Show that dE/dt is the rate at which energy is dissipated by $F_{\text{damp.}}$.

2.) The potential energy of a one-dimensional mass m at a distance r from the origin is

$$U(r) = U_0\Big(\frac{r}{R} + \lambda^2\,\frac{R}{r}\Big)$$

for $0 < r < \infty$, with U_0, R and λ all positive constants.

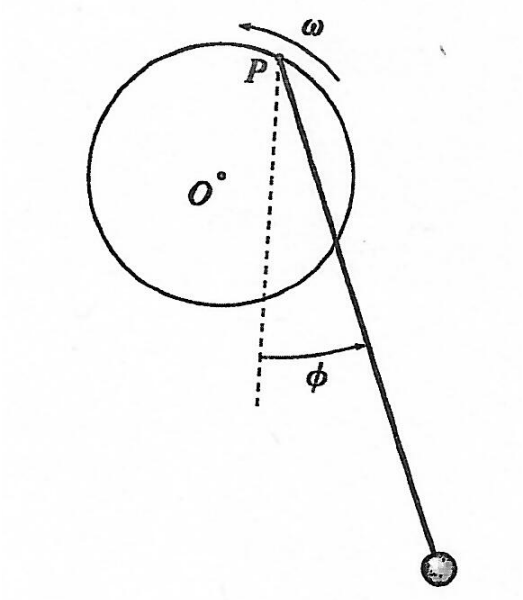
a.) Find the equilibrium position r_0.

b.) Let x be the distance from equilibrium and show that, for small x, the PE has the form $U = \text{const} + \frac{1}{2}\,k\,x^2$.

c.) What is the angular frequency of small oscillations?

3.) Write down the Lagrangian for a cylinder (mass m, radius R, and moment of inertia I) that rolls without slipping straight down an inclined plane which is at an angle α from the horizontal. Use as your generalized coordinate the cylinder's distance x measured down the plane from its starting point. Write down the Lagrange equation and solve it for the cylinder's acceleration $\ddot{x}$. Remember that $T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$, where v is the velocity of the center of mass and ω is the angular velocity.

4.) The figure below shows a simple pendulum (mass m, length ℓ) whose point of support P is attached to the edge of a wheel (center O, radius R) that is forced to rotate at a fixed angular velocity ω. At $t = 0$, the point P is level with O on the right.

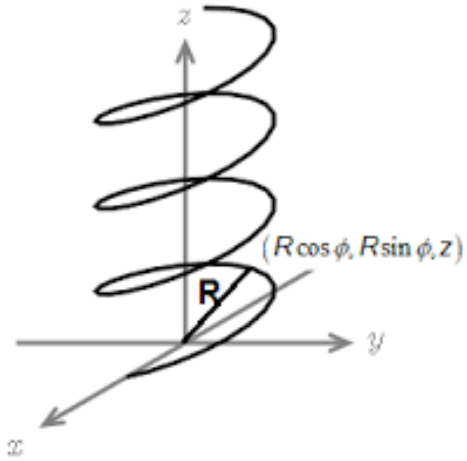


a.) Write down the Lagrangian for the system. *[Hint: Be careful with the kinetic energy T. A safe way to get the velocity right is to write down the position of the bob at time t, and then differentiate.]*

b.) Derive the equation of motion for the angle ϕ.

c.) Does your answer make sense in the limit $\omega \to 0$? Explain your reasoning.

5.) A smooth wire is bent into the shape of a helix, with cylindrical polar coordinates $\rho = R$ and $z = \lambda\phi$, where R and λ are constants and the z axis is vertically up (and gravity vertically down).

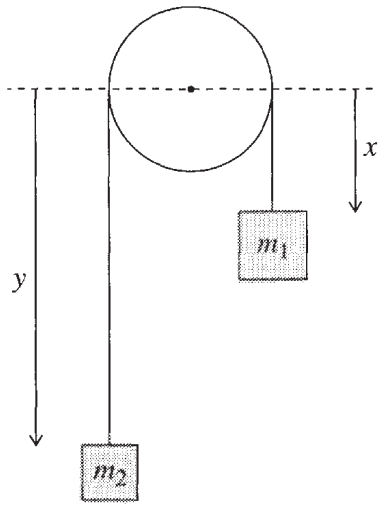


a.) Using z as your generalized coordinate, write down the Lagrangian for a bead of mass m threaded on the wire.

b.) Find the Lagrange equation and hence the bead's vertical acceleration $\ddot{z}$.

c.) In the limit that $R \to 0$, what is $\ddot{z}$?

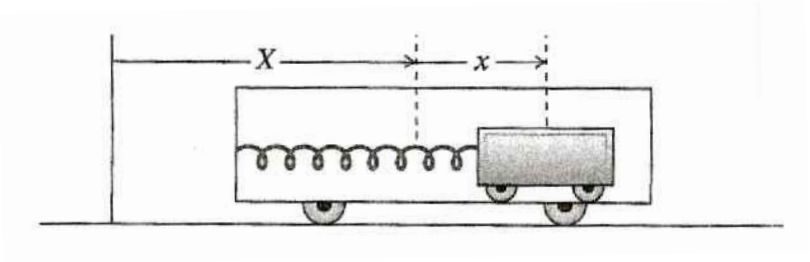
6.) Consider the simple Atwood machine shown below.



a.) Write down the Lagrangian of the system in terms of a suitably chosen generalized coordinate. Take the pulley to have radius R, and include the effects of its moment of inertia I.

b.) Use the Lagrangian to find the acceleration of the system.

7.) A small cart (mass m) is mounted on rails inside a large cart. The two are attached by a spring (force constant k) in such a way that the small cart is in equilibrium at the midpoint of the large. The distance of the small cart from its equilibrium is denoted x and that of the large one from a fixed point on the ground is X, as shown below.



The large cart is now forced to oscillate such that $X = A\cos\omega t$, with both A and ω fixed. Set up the Lagrangian for the motion of the small cart and show that the Lagrange equation has the form

$$\ddot{x} + \omega_0^2 x = B\cos\omega t$$

where ω_o is the natural frequency $\omega_0 = \sqrt{k/m}$ and B is a constant.

8.) Calculate the motion followed by a mass dropped from rest at the surface of a uniform density planet mass M. through a hole that extends through the center of the earth through to the other side of the planet. Ignore air resistance, the rotation of the planet. Give the general formula in terms of Newton's constant, the radius, and the mass of the planet for the time it takes the object to reach the other side.

9.) A rifle fires a bullet straight up with velocity v_0. The bullet rises to a certain height, and then falls back to the ground.

a.) In the case of no air resistance, determine the height H_0 and the speed of the bullet, v_{ground}, when it returns.

b.) Now add air resistance assuming the magnitude of the force is $F = m\alpha v^2$, where v is the velocity. Consider the upward trajectory, and express the equation of motion terms of the variable $u(x) = v^2(x)$.

c.) Solve the equation of motion to determine the height H that the bullet reaches in the presence of air resistance.

d.) Show that in the limit $\alpha \to 0$, $H \to H_0$.

10.) A particle of mass m is performing one-dimensional motion subject to the force function $F(x) = -F_0 \sin(cx)$. At $t = 0$ the particle's position is $x(0) = x_0 = 0$ and its velocity is $v(0) = v_0$.

a.) Find the potential energy as a function of x and sketch it.

b.) Find the velocity as a function of x.

c.) Find the condition on the initial velocity v_0 for which the motion is periodic.

d.) Find the turning points for the periodic motion.

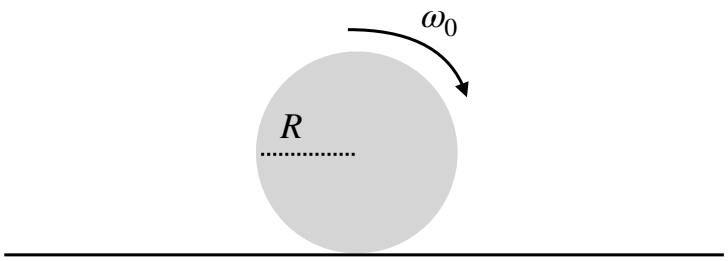
e.) Suppose that v_0 is so small that the periodic motion can be treated as a harmonic oscillator. Find the period of these oscillations and the oscillation amplitude.

11.) A rigid body consists of three equal masses (m) fastened at the positions $(a, 0, 0), (0, a, 2a)$, and $(0, 2a, a)$.

a.) Find the inertia tensor I.

b.) Find the principal moments and a set of orthogonal principal axes.

12.) A billiard ball of radius R has its center of mass at rest, but is spinning about a horizontal axis with angular speed ω_0 as shown in the figure below. The coefficient of sliding friction between the ball and the table on which it rests is μ. At $t = 0$, the ball starts moving to the right, with combined linear and rotational motion.



a.) What are the forces and torques on the system?

b.) Determine the time T when the ball rolls without slipping, $v(T) = R\,\omega(T)$.

c.) How far has the ball traveled in this time?

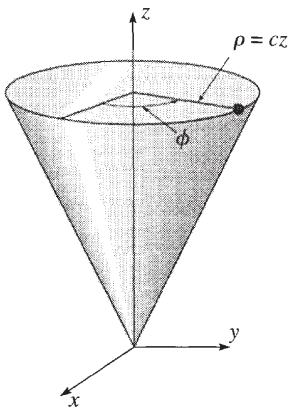
13.) The center of a long frictionless rod is pivoted at the origin and the rod is forced to rotate at a constant angular velocity Ω in a horizontal plane. Write down the equation of motion for a bead that is threaded on the rod, using the coordinates x and y of a frame that rotates with the rod (with x along the rod and y perpendicular to it). Solve for $x(t)$. What is the role of the centrifugal force? What of the Coriolis force?

14.) A high-speed train is traveling at a constant 150 m/s (about 300mph) on a straight, horizontal track across the South Pole. Given the Coriolis force, find the angle between a plumb line suspended from the ceiling inside the tram and another inside a hut on the ground. In what direction is the plumb line on the train deflected?

15.) A bead of mass m is threaded on a frictionless wire that is bent into a helix with cylindrical polar coordiuates (ρ, ϕ, z) satisfying $z = c\phi$ and $\rho = R$, with c and R constants. The z axis points vertically up and gravity vertically down. Using ϕ as your generalized coordinate, write down the kinetic and potential energies, and hence the Hamiltonian $\mathcal{H}$ as a function of ϕ and its conjugate momentum p. Write down Hamilton's equations and solve for $\ddot{\phi}$ and hence $\ddot{z}$. Explain your result in terms of Newtonian mechanics and discuss the special case that $R = 0$.

16.) Consider a particle of mass m moving in two dimensions, subject to a force $\mathbf{F} = -kx\hat{\mathbf{x}} + K\hat{\mathbf{y}}$, where k and K are positive constants. Write down the Hamiltonian and Hamilton's equations, using x and y as generalized coordinates. Solve the latter and describe the motion.

17.) Consider a point particle with mass m constrained to move on the frictionless surface of a vertical cone defined in terms of cylindrical polar coordinates (ρ, ϕ, z) by the constraint $\rho = c\,z$ and $z > 0$. The particle is in a uniform gravitational field with $\vec{g} = -g\hat{z}$.



a.) Write down the Hamiltonian of the system in terms of the cylindrical polar coordinates (ρ, ϕ, z). Are there any *ignorable/cyclic* variables?

b.) Derive Hamilton's equations of motion. Point out constants of motion (conserved quantities) you encounter and the corresponding conservation laws.

c.) Use the connection between the Hamiltonian and the energy E to show that for any given solution there are maximum and minimum heights z_{max} and z_{min} between which motion is confined. Use this result to describe the motion of the mass on the cone.

18.) A planet of mass m is orbiting a star of mass M. The planet experiences a small drag force $\mathbf{F} = -\alpha\mathbf{v}$ due to its motion through the star's dense atmosphere. Assuming a circular orbit with radius $r = r_0$ at $t = 0$, calculate the time dependence of the orbit. (The motion is actually a spiral, but you may make approximations assuming $\dot{r}$ is small).

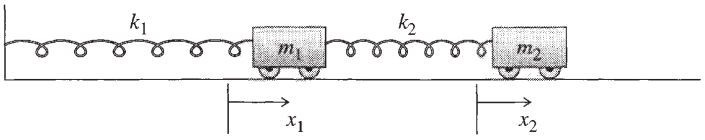
19.) Consider a planet orbiting the fixed sun. Take the plane of the planet's orbit to be the $\hat{x}\hat{y}$ plane, with the sun at the origin, and label the planet's position by polar coordinates (r, ϕ).

a.) Show that the planet's angular momentum $\ell = m\, r^2\, \omega$, where $\omega = \dot{\phi}$ is the planet's angular velocity about the sun.

b.) Show that the rate at which the planet 'sweeps out area' (meaning the area dA in the plane covered by the position vector $\vec{r}$ in a time dt) is

$$\frac{dA}{dt} = \frac{1}{2}\, r^2\, \omega$$

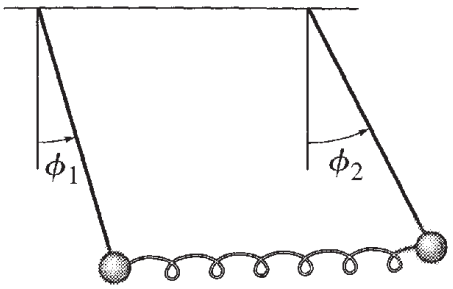
and hence that $\frac{dA}{dt} = \frac{\ell}{2m}$. Deduce Kepler's second law.



20.) a.) Find the normal frequencies, ω_1 and ω_2, for the two carts shown above, assuming that $m_1 = m_2$ and $k_1 = 3k_2/2$.

b.) Find and describe the motion for each of the normal modes in turn.

21.) Consider two identical plane pendulums (each of length L and mass m) that are joined by a massless spring (force constant k) as shown in the figure below. The pendulums' positions are specified by the angles ϕ_1 and ϕ_2 shown. The natural length of the spring is equal to the distance between the two supports, so the equilibrium position is at $\phi_1 = \phi_2 = 0$ with the two pendulums vertical.



a.) Write down the total kinetic energy and the gravitational and spring potential energies. [Assume that both angles remain small at all times. This means that the extension of the spring is well approximated by $L\,(\phi_2 - \phi_1)$.] Write down the Lagrange equations of motion.

b.) Find and describe the normal modes for these two coupled pendulums.

22.) a.) Using elementary Newtonian mechanics find the period of a mass m_1 in a circular orbit of radius r around a fixed mass m_2.

b.) Using the separation into CM and relative motions, find the corresponding period for the case that m_2 is not fixed and the masses circle each other a constant distance r apart. Discuss the limit of this result if $m_2 \to \infty$.

c.) What would be the orbital period if the earth were replaced by a star of mass equal to the solar mass, in a circular orbit, with the distance between the sun and star equal to the present earth-sun distance? (The mass of the sun is more than 300,000 times that of the earth.)

23.) Conider the following cycle for an ideal monoatomic gas. Starting from an initial pressure and an initial volume (P_1, V_1) we perform a compression at constant volume (P_2, V_2), then an expansion at constant pressure (P_3, V_3), followed by an adiabatic expansion (P_4, V_4) and finishing with an isothermal compression.

a.) Draw the PV diagram

b.) Is the pressure P_4 bigger or smaller than P_1 ? Explain your reasoning.

c.) Calculate W, Q and ΔU for each of the four processes.

24.) Starting with the multiplicity of an Einstein Solid:

$$\Omega(q, N) = \begin{pmatrix} q + N - 1 \\ q \end{pmatrix}$$

a.) Calculate the entropy as a function of the energy U and N.

b.) Calculate the temperature as a function of U and N.

c.) Calculate the heat capacity. Explain the behavior at low and high temperature.

25.) Lets imagine an ideal monoatomic gas that undergoes a Carnot cycle:

a.) Explain each of the steps and draw the PV diagram.

b.) For each of the steps calculate $\Delta U, Q, W$ and the efficiency of the engine.

c.) Draw the ST diagram.

26.) a.) Find a relation between the heat capacity at constant V and at constant P (C_V vs. C_P). Verify that it is valid for an ideal gas. *Hint: Write the entropy as a generic function of T and P and then suppose that P is a function of T and V*

b.) Find a relation between κ_T (compressibility at constant T) and κ_S (compressibility at constant S). Verify that it is valid for an ideal gas. *Hint: You can use a similar approach as the previous one*

c.) How does the temperature change with pressure in the throttling process? Apply the formula for an ideal monoatomic gas.

27.) A cylindrical container of initial volume V_0 contains N atoms of a classical ideal gas at temperature T. One end of the container is movable, and so we can compress the gas slowly, reducing the volume of the gas by two percent while keeping the temperature of the gas the same.

a.) What is the change in entropy of the confined gas?

b.) How much work do we do in compressing the gas?

c.) How much energy was absorbed by the environment (through heating)?

d.) What was the entropy change to the environment?

28.) A copper penny, initially at temperature T_i is placed in contact with a large block of ice that serves as a heat reservoir and has a constant temperature T_{res} (well below freezing). Take the penny's heat capacity to have the constant value C, and specify $T_i \neq T_{res}$ (by a finite amount). The following questions pertain after the joint system has come to thermal equilibrium.

a.) What are the entropy changes of the penny and of the ice block?

b.) What sign does the total change in entropy have [according to your calculations in part a.)]?

c.) Is the sign independent of whether the penny was hotter or colder than the ice block? Explain your answer.

29.) Two identical bubbles of gas form at the bottom of a lake, then rise to the surface. Because the pressure is much lower at the surface than at the bottom, both bubbles expand as they rise. However, bubble A rises very quickly so that no heat is exchanged between it and the water. Meanwhile, bubble B rises slowly (impeded by a tangle of seaweed), so that it always remains in thermal equilibrium with the water (which you can assume has the same temperature everywhere).

a.) How does the temperature of the bubbles change as they rise in the water?

b.) How does the entropy change? Explain your reasoning.

c.) Which of the two bubbles is larger by the time they reach the surface? Explain your reasoning.

30.) Imagine some helium in a cylinder with an initial volume of 1 liter and an initial pressure of 1 atm. Somehow the helium is made to expand to a final volume of 3 liters, in such a way that its pressure rises in direct proportion to its volume.

a.) Sketch a graph of pressure vs. volume for this process.

b.) Calculate the work done on the gas during this process, assume that there are no 'other' types of work being done.

c.) Calculate the change in the helium's energy content during this process.

d.) Calculate the amount of heat added or removed from the helium during this process.

31.) a.) Which of the following expressions:

$$S = Nk \sin\left(\frac{U^{3/2}V}{\alpha N^{5/2}}\right) \quad S = Nk\left(\log\left(\frac{U^{3/2}V}{\alpha N^{5/2}}\right) + \frac{5}{2}\right) \quad S = S_0 - Nk\frac{U^{3/2}V}{\alpha N^{5/2}}$$

could be interpreted as entropy? Explain your answer.

b.) Calculate the units of α.

c.) For the expression(s) that can be interpreted as entropy, calculate P, T and μ

32.) Given the following Gibbs free energy:

$$G = -k\,T N\,\log\left(\frac{a\,T^{5/2}}{P}\right)$$

where a is a constant (with dimension to make the argument of the logarithm dimensionless). Recall that $G(p,T) = U + PV - TS$. Compute:

a.) the entropy,

b.) the heat capacity at constant pressure C_P,

c.) the equation of state (the connection among V, P, N and T),

d.) the average energy $\langle E \rangle$.