

QUANTUM MECHANICS PRELIMINARY EXAM JANUARY 2025 TEST QUESTION BANK

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Note: The preliminary examination problems will be drawn from this set. While the spirit and methodology of the problems will be unchanged, some specifics (an initial wave function, what expectation values are asked for, etc.) may change.

1.) Time Dependence: Show that in one dimension

$$\frac{d}{dt} \int_{-\infty}^{+\infty} \Psi_1^* \Psi_2 dx = 0$$

for any two normalizable solutions to the Schrödinger equation

- 2.) Prove the following relation between the uncertainty in position and the uncertainty in total energy :

$$\sigma_x \sigma_H \geq \frac{\hbar}{2m} |\langle p \rangle|.$$

Why does this not tell us much for bound state (normalizable) stationary states (in the center of mass reference frame)?

- 3.) Use Ehrenfest's theorem, and appropriate choices for the operators that appear in the theorem, to (a) prove classical energy conservation; (b) prove that $\langle p \rangle = m\langle v \rangle$; and prove Newton's second law: $\langle F \rangle = d\langle p \rangle/dt$.

4.) A free particle has the initial wave function

$$\psi(x, 0) = Ae^{-a|x|}$$

where A and a are positive real constants.

- a.) Normalize $\psi(x, 0)$
- b.) find the momentum wave function $\Phi(k)$
- c.) Construct $\psi(x, t)$ in the form of an integral.

5.) Consider the wavefunction

$$\psi(x, t) = Ae^{-\lambda|x|}e^{-i\omega t}$$

where A , λ and ω are positive real constants.

- a.) Normalize ψ
- b.) Determine the expectation value of x and x^2
- c.) Find the standard deviation (σ) of x . Sketch the graph of $|\psi|^2$, as a function of x and mark the points $(\langle x \rangle - \sigma)$ and $(\langle x \rangle + \sigma)$. What is the probability the particle would be found in this range? You may leave your answer in integral form

- 6.) Calculate $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$, $\langle p^2 \rangle$, σ_x and σ_p for the n th stationary state of the infinite square well. Prove that the uncertainty relationship is satisfied.

- 7.) At $t = 0$ the wavefunction of a particle in a one-dimensional, infinitely-deep box with walls at $x = 0$ and $x = a$ is given by

$$\psi(x, 0) = C \sin(2\pi x/a) \cos(\pi x/a), \quad (1)$$

where C is a constant.

- a.) Find $\psi(x, t)$ for $t > 0$. [*Hint: use trig identities to simplify the initial wave function into a sum of stationary states.*]
- b.) What is the energy expectation value for $t > 0$? Express answers for parts (a) and (b) in terms of E_n ($n = 1, 2, \dots$), the energies of the stationary states.
- c.) Find the probability P_n that the measurement of the particle's energy at the time t finds the value E_n .

8.) A particle of mass m sits in a finite square well of width $2a$ and depth V_0 :

$$V(x) = \begin{cases} 0 & |x| > a \\ -V_0 & -a \leq x \leq a \end{cases}$$

a.) Consider the bound-state problem ($E < 0$). Defining

$$k^2 = \frac{2m|E|}{\hbar^2}, \quad \ell^2 = \frac{2m}{\hbar^2}(V_0 - |E|) \quad (2)$$

solve the Schrodinger equation in all regions. As the potential is symmetric, the solutions can be grouped into even and odd.

- b.) Focusing on the **odd** solutions from part a.), find the equation that determines the allowed energy eigenvalues. Express your result as a transcendental equation in terms of $z = \ell a$ and $z_0 = (\sqrt{2mV_0/\hbar^2})a$.
- c.) The depth of the potential is controlled by $V_0 \sim z_0^2$. In the limit that we make the potential very shallow, is there always an odd bound state? Explain why or why not. *[Hint: graph both sides of the transcendental equation]*

- 9.) An electron in a one dimensional infinitely deep box with wall at $x = 0$ and $x = a$ is in the quantum state:

$$\begin{aligned}\Psi(x) &= A & 0 < x < \frac{a}{2} \\ \Psi(x) &= -A & \frac{a}{2} < x < a\end{aligned}\tag{3}$$

- a.) Obtain an expression for the normalization constant A .
- b.) If the electron is in this state, what is the lowest energy it can be measured to have?

- 10.) a.) Write down the Hamiltonian, eigenenergies and eigenstates for a two dimensional harmonic oscillator with distinct spring constants k_x and k_y .
b.) If $k_y = 4k_x$, show that the eigenenergies may be written as:

$$E_{n_1 n_2} = \hbar\omega_0 \left(n_1 + 2n_2 + \frac{3}{2} \right) \quad (4)$$

- c.) What is the order of degeneracy of $E_{2,3}$? List the corresponding eigenstates.

11.) A particle with mass m in a harmonic oscillator potential is in an initial state

$$\Psi(x, 0) = A \left(1 - 2\sqrt{\frac{m\omega}{\hbar}} x \right)^2 \exp \left[-\frac{m\omega}{2\hbar} x^2 \right],$$

where A is a normalization constant.

- a.) What is the expectation value of the energy?
- b.) At some later time t_0 the wave function is

$$\Psi(x, t_0) = B \left(1 + 2\sqrt{\frac{m\omega}{\hbar}} x \right)^2 \exp \left[-\frac{m\omega}{2\hbar} x^2 \right].$$

What is the smallest possible value of t_0 ?

12.) A particle in a harmonic oscillator potential,

$$V(x) = \frac{1}{2}m\omega^2 x^2$$

starts out ($t = 0$) in the state $A(4|2\rangle + 3|3\rangle)$.

- a.) If you measured the energy of this particle, what values might you get and with what probabilities?
- b.) Find the expectation value $\langle x \rangle$, as a function of time.
- c.) Find the expectation value $\langle p \rangle$, as a function of time.
- d.) Check that Ehrenfest's theorem,

$$\frac{d\langle p \rangle}{dt} = \left\langle -\frac{\partial V}{\partial x} \right\rangle,$$

holds for this state.

- 13.) A harmonic oscillator potential is in a state such that a measurement of the energy would yield either $\frac{1}{2}\hbar\omega$ or $\frac{3}{2}\hbar\omega$ with equal probability.
- a.) What is the largest possible value of $\langle p \rangle$ in such a state?
 - b.) If this maximal value occurs at $t = 0$, what is $\Psi(x, t)$?

14.) An electron in a Hydrogen atom is in the state:

$$|\psi\rangle = \frac{4}{5}|1, 0, 0\rangle + \frac{3i}{5}|2, 1, 1\rangle$$

where the indices stand for $|n, \ell, m\rangle$, the usual energy eigenstates of hydrogen.

- a.) What is $\langle E \rangle$ for this state? What are $\langle L^2 \rangle$ and $\langle L_z \rangle$?
- b.) How do the expectation values in part a.) depend on time?

- 15.) a.) Calculate $\langle r \rangle$ and $\langle r^2 \rangle$ for an electron in the ground state of hydrogen.
- b.) Find $\langle x \rangle$ and $\langle x^2 \rangle$ for an electron in the ground state of hydrogen. Hint: If you exploit the symmetry of the ground state, this does not require any additional integrations.
- c.) Find $\langle x^2 \rangle$ in the ψ_{211} state.

Express all your results in terms of the Bohr radius.

- 16.) What is the most probable value of r , in the ground state of hydrogen? (The answer is not zero!) *Hint: First you must figure out the probability that the electron would be found between r and $r + dr$.*

17.) An electron in a Hydrogen atom is in a state described by the wave function

$$\psi(\mathbf{r}) = \frac{1}{2}\sqrt{2}[\psi_{n00}(\mathbf{r}) - \psi_{210}(\mathbf{r})], \quad (5)$$

where $\psi_{n\ell m}(\mathbf{r})$ is the wave function of a stationary state with principal quantum number n , angular momentum quantum number ℓ , and its projection m .

- a.) Is this state an eigenstate of the Hamiltonian if $n = 1$? If $n = 2$? Explain.
- b.) Find the expectation value of the energy for $n = 1$ and for $n = 2$. Express your answers in terms of the energy of the ground state, $E_1 = -R$, where R is the Rydberg constant.
- c.) Find the expectation value of $\mathbf{L}^2$.
- d.) Suppose $n = 2$. Find the expectation value of $\mathbf{r} = (x, y, z)$ using the following information: $\int z\psi_{210}(\mathbf{r})\psi_{200}(\mathbf{r})d^3r = -3a_0$ (a_0 is the Bohr radius).

- 18.) a.) A particle of spin 1 and a particle of spin 2 are at rest in a configuration such that the total spin is 3, and its z -component is $-3\hbar$. If you measured the z -component of the angular momentum of the spin-2 particle, what values might you get?
- b.) Repeat part (a), but for the z -component being $\hbar$.
- c.) An electron with spin down is in the $\psi_{nlm} = \psi_{510}$ state of the hydrogen atom. If you could measure the total angular momentum squared of the electron, what values might you get?

The diagram illustrates the addition of angular momentum for two particles. It shows the possible values of the total angular momentum J and its z -component M for different combinations of the individual angular momenta J_1 and J_2 . The tables are arranged in a way that shows the step-by-step addition of the angular momenta, starting from the individual states and leading to the final combined states.

- 19.) a.) Starting with the canonical commutator relations for position and momentum, show the following:

$$\begin{array}{lll} [L_z, x] & = & i\hbar y, & [L_z, y] & = & -i\hbar x, & [L_z, z] & = & 0, \\ [L_z, p_x] & = & i\hbar p_y, & [L_z, p_y] & = & -i\hbar p_x, & [L_z, p_z] & = & 0. \end{array}$$

- b.) Use these relations to obtain $[L_z, L_x] = i\hbar L_y$.
- c.) Evaluate $[L_z, r^2]$ and $[L_z, p^2]$.
- d.) Show that the Hamiltonian $H = p^2/2m + V$ commutes with all three components of $\mathbf{L}$, provided that V only depends on r .

20.) An electron is in the spin- $\frac{1}{2}$ state

$$\chi = A \begin{pmatrix} 3i \\ 4 \end{pmatrix}.$$

- a.) Determine the normalization constant A .
- b.) Find the expectation values of S_x , S_y , S_z .
- c.) Find the uncertainties σ_{S_x} , σ_{S_y} , and σ_{S_z} .

- 21.) A particle moving on a sphere is described by the wavefunction (in spherical coordinates)

$$\psi = C \sin \theta \sin \phi, \quad (6)$$

where C is a constant.

- a.) What is the expectation value of L_z for this state?
- b.) What are the possible outcomes of the measurements of L_z , and what are their probabilities?
- c.) What is the expectation value of L_x ?
- d.) What is the expectation value of L^2 ?
- e.) Find the normalization constant C , and the expectation values of $\cos \theta$ and $\cos^2 \theta$.

- 22.) A particle with spin 1 is in the state with the spin projection on the z axis $S_z = \hbar$.
- a.) What are the expectation values of S_x^2 and S_y^2 for this state?
 - b.) Suppose S_x is measured. What are the possible outcomes of this measurement and what are the corresponding probabilities?

- 23.) A particle at rest with spin $1/2$ and a gyromagnetic ratio γ is placed in a magnetic field $\vec{B}$ directed along the x axis.
- a.) Write down the Schrodinger equation describing the evolution of the particle's wavefunction.
 - b.) Solve the equation assuming that at $t = 0$ the particle is in an eigenstate of $\hat{S}_z$ with eigenvalue $\hbar/2$.
 - c.) Calculate the expectation value of $\hat{S}_z$ and S_y as functions of t

- 24.) a.) Suppose $f(x)$ and $g(x)$ are two eigenfunctions of an operator $\hat{Q}$ with the same eigenvalue q . Show that any linear combination of f and g is itself an eigenfunction of $\hat{Q}$ with eigenvalue q
- b.) Check that $f(x) = e^x$ and $g(x) = e^{-x}$ are eigenfunctions of the operator d^2/dx^2 , with the same eigenvalue. Construct two linear combinations of f and g that are orthogonal eigenfunctions on the interval $(-1, 1)$.

- 25.) An operator $\hat{A}$ has two normalized eigenstates ψ_1 and ψ_2 , with eigenvalues a_1 and a_2 . Likewise, operator $\hat{B}$ has two normalized eigenstates ϕ_1 and ϕ_2 , with eigenvalues b_1 and b_2 . The eigenstates are related by

$$\psi_1 = (3\phi_1 + 4\phi_2)/5, \quad \psi_2 = (4\phi_1 - 3\phi_2)/5.$$

- a.) Observable A (associated with the operator $\hat{A}$) is measured, and the value a_1 is obtained. What is the state of the system immediately after this measurement?
- b.) If observable B is now measured, what are the possible outcomes and their probabilities?
- c.) Right after the measurement of B , A is measured again. What is the probability of getting a_1 ? Note that the outcome of the measurement of B is unknown!

- 26.) Consider a three-dimensional vector space spanned by an orthonormal basis $|1\rangle, |2\rangle, |3\rangle$. Kets $|\alpha\rangle$ and $|\beta\rangle$ are given by:

$$\begin{aligned} |\alpha\rangle &= i|1\rangle - 2|2\rangle - i|3\rangle \\ |\beta\rangle &= i|1\rangle + 2|3\rangle \end{aligned} \tag{7}$$

- a.) Construct $\langle\alpha|$ and $\langle\beta|$ in terms of the dual basis $\langle 1|, \langle 2|, \langle 3|$.
- b.) Find $\langle\alpha|\beta\rangle$ and $\langle\beta|\alpha\rangle$ and confirm that $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$.
- c.) Find all nine matrix elements of the operator $\hat{A} = |\alpha\rangle\langle\beta|$ in this basis and construct the matrix A . Is it Hermitian?

27.) a.) Prove the following commutator identity:

$$[AB, C] = A[B, C] + [A, C]B.$$

b.) Show that

$$[x^n, p] = i\hbar nx^{n-1}.$$

c.) Starting from more basic commutators, show that for any function $f(x)$,

$$[f(x), p] = i\hbar \frac{df}{dx}$$

Although this equation is given on the equation sheet, repeating it without any derivation is not a sufficient answer.

d.) Show that if $[A, [A, B]] = 0$ then

$$[f(A), B] = [A, B] (df/dA).$$

Here the function of an operator $f(A)$ should be understood as a power series: $f(A) = \sum_n a_n A^n = a_0 + a_1 A + a_2 A^2 + a_3 A^3 + \dots$

[Hint: if you are stuck in b.) - d.), try applying the commutators to a test function of x .]

28.) The Hamiltonian for a certain three-level system is represented by the matrix

$$\mathbf{H} = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

with a positive frequency ω . Two other observables, A and B , are represented by

$$\mathbf{A} = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \mathbf{B} = \mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

where λ and μ are real and positive.

- a.) Find the eigenvalues and normalized eigenvectors of $\mathbf{H}$, $\mathbf{A}$, and $\mathbf{B}$.
- b.) Suppose the system starts out in the generic state

$$|\mathcal{S}(0)\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

with $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$. Find the expectation value of H , A , and B at $t = 0$.

- c.) Find $|\mathcal{S}(t)\rangle$. If you measured the energy of this state at some time $t > 0$, what values might you get, and what is the probability of each? Answer the same questions for measuring A and B .

29.) Neutrinos are uncharged, relativistic particles produced in processes such as

$$\begin{aligned} p &\rightarrow n + e^+ + \nu_e \\ \pi^+ &\rightarrow \mu^+ + \nu_\mu \end{aligned}$$

describing respectively a proton decaying to a neutron, a positron and an electron neutrino (ν_e), and a pion decaying to a muon and a muon neutrino (ν_μ). The quantum states $|\nu_e\rangle$ and $|\nu_\mu\rangle$ are flavor eigenstates to the Hamiltonian describing their creation in processes such as the ones listed above (weak interaction). However, when neutrinos propagate in free space, the only Hamiltonian of relevance is that due to the (relativistic) energy of the particles. The eigenstates of this Hamiltonian are referred to as the mass eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$, with energies E_1 and E_2 ($E_1 \neq E_2$). Either of the two bases can be used to express any general state in this system. Assume that the relation between the bases is given by

$$\begin{aligned} |\nu_1\rangle &= \cos \frac{\theta}{2} |\nu_e\rangle + \sin \frac{\theta}{2} |\nu_\mu\rangle \\ |\nu_2\rangle &= \sin \frac{\theta}{2} |\nu_e\rangle - \cos \frac{\theta}{2} |\nu_\mu\rangle \end{aligned}$$

where the θ is generally referred to as the mixing angle.

- a.) Find expressions for $|\nu_e\rangle$ and $|\nu_\mu\rangle$ in terms of the mass eigenstates and the mixing angle.

Assume that an electron neutrino is created at $t = 0$.

- b.) Provide an expression for the neutrino state $|\psi(t)\rangle$ as it propagates through free space towards a detector. Give this in terms of the mass eigenstates and energies, and the mixing angle.
- c.) Show that the probability that a measurement at time t would detect a muon neutrino is given by

$$\mathcal{P}_{\nu_e \rightarrow \nu_\mu} = |\langle \nu_\mu | \psi(t) \rangle|^2 = \sin^2 \theta \sin^2 \left(\frac{(E_1 - E_2)t}{2\hbar} \right)$$

- 30.) Consider two non-interacting spin-less particles, each with mass m , in the infinite square well. One is in the ground state and the other in the first excited state. Calculate $\langle (x_1 - x_2)^2 \rangle$ when the particles are:
- a.) Distinguishable.
 - b.) Identical with a symmetric wave function.
 - c.) Identical with an antisymmetric wave function.